مسألة اليوم صفحة 47

$$A = \int_1^2 \frac{1}{x^3 + x}\, dx$$

لإيجاد قيمة هذا التكامل نجزئ المقدار $\frac{1}{x^3+x}$ إلى كسور جزئية يمكن إيجاد تكاملاتها بسهولة كما يأتي:

$$\frac{1}{x^3 + x} = \frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow 1 = A(x^2 + 1) + (Bx + C)(x)$$

$$x = 0 \Rightarrow A = 1$$

$$x = 1 \Rightarrow 1 = 2A + B + C \Rightarrow 1 = 2 + B + C$$

$$x = -1 \Rightarrow 1 = 2A + B - C \Rightarrow 1 = 2 + B - C$$

بحل هاتين المعادلتين نجد أنّ: $C = 0$ ، و $B = -1$

$$A = \int_1^2 \frac{1}{x^3 + x}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{-x}{x^2 + 1}\right) dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2 + 1|\Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 - \ln 1 + \frac{1}{2}\ln 2$$

$$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5 \quad = \frac{1}{2}\ln\frac{8}{5}$$

أتحقق من فهمي صفحة 49	
a	$\frac{x-7}{x^2-x-6}=\frac{x-7}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$ $\Rightarrow x-7=A(x+2)+B(x-3)$ $x=3\Rightarrow A=-\frac{4}{5}$ $x=-2\Rightarrow B=\frac{9}{5}$ $\int\frac{x-7}{x^2-x-6}dx=\int\left(\frac{-\frac{4}{5}}{x-3}+\frac{\frac{9}{5}}{x+2}\right)dx$ $=-\frac{4}{5}\ln\lvert x-3\rvert+\frac{9}{5}\ln\lvert x+2\rvert+C$
b	$\frac{3x-1}{x^2-1}=\frac{3x-1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$ $\Rightarrow 3x-1=A(x+1)+B(x-1)$ $x=1\Rightarrow A=1$ $x=-1\Rightarrow B=2$ $\int\frac{3x-1}{x^2-1}dx=\int\left(\frac{1}{x-1}+\frac{2}{x+1}\right)dx$ $=\ln\lvert x-1\rvert+2\ln\lvert x+1\rvert+C$

أتحقق من فهمي صفحة 51	
a	$\frac{x+4}{(2x-1)(x-1)^2}=\frac{A}{2x-1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ $\Rightarrow x+4=A(x-1)^2+B(2x-1)(x-1)+C(2x-1)$ $x=\frac{1}{2}\Rightarrow A=18$ $x=1\Rightarrow C=5$ $x=0\Rightarrow 4=A+B-C\Rightarrow B=-9$ $\int\frac{x+4}{(2x-1)(x-1)^2}dx=\int\left(\frac{18}{2x-1}+\frac{-9}{x-1}+\frac{5}{(x-1)^2}\right)dx$ $=\frac{18}{2}\ln\|2x-1\|-9\ln\|x-1\|-\frac{5}{x-1}+C$ $=9\ln\|2x-1\|-9\ln\|x-1\|-\frac{5}{x-1}+C$
b	$\frac{x^2-2x-4}{x^3-4x^2+4x}=\frac{x^2-2x-4}{x(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x}$ $\Rightarrow x^2-2x-4=Ax(x-2)+Bx+C(x-2)^2$ $x=2\Rightarrow B=-2$ $x=0\Rightarrow C=-1$ $x=1\Rightarrow -5=-A+B+C\Rightarrow A=2$ $\int\frac{x^2-2x-4}{x^3-4x^2+4x}dx=\int\left(\frac{2}{x-2}+\frac{-2}{(x-2)^2}+\frac{-1}{x}\right)dx$ $=2\ln\|x-2\|+\frac{2}{x+2}-\ln\|x\|+C$

أتحقق من فهمي صفحة 52	
a	$\frac{3x+4}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$ $\Rightarrow 3x+4 = A(x^2+4) + (Bx+C)(x-3)$ $x = 3 \Rightarrow A = 1$ $x = 0 \Rightarrow 4 = 4A - 3C \Rightarrow C = 0$ $x = 1 \Rightarrow 7 = 5A - 2B - 2C \Rightarrow B = -1$ $\int \frac{3x+4}{(x-3)(x^2+4)} dx = \int \left(\frac{1}{x-3} - \frac{x}{x^2+4}\right) dx$ $= \int \left(\frac{1}{x-3} - \frac{1}{2} \times \frac{2x}{x^2+4}\right) dx$ $= \ln\|x-3\| - \frac{1}{2}\ln\|x^2+4\| + C$
b	$\frac{7x^2-x+1}{x^3+1} = \frac{7x^2-x+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ $\Rightarrow 7x^2-x+1 = A(x^2-x+1) + (Bx+C)(x+1)$ $x = -1 \Rightarrow A = 3$ $x = 0 \Rightarrow 1 = A + C \Rightarrow C = -2$ $x = 1 \Rightarrow 7 = A + 2B + 2C \Rightarrow B = 4$ $\int \frac{7x^2-x+1}{x^3+1} dx = \int \left(\frac{3}{x+1} + \frac{4x-2}{x^2-x+1}\right) dx$ $= \int \left(\frac{3}{x+1} + 2 \times \frac{2x-1}{x^2-x+1}\right) dx$ $= 3\ln\|x+1\| + 2\ln\|x^2-x+1\| + C$

أتحقق من فهمي صفحة 53	
a	$\int \frac{4x^3-5}{2x^2-x-1}dx = \int\left(2x+1+\frac{3x-4}{2x^2-x-1}\right)dx$ $\frac{3x-4}{2x^2-x-1} = \frac{3x-4}{(2x+1)(x-1)} = \frac{A}{2x+1}+\frac{B}{x-1}$ $\Rightarrow 3x-4 = A(x-1)+B(2x+1)$ $x=-\frac{1}{2} \Rightarrow A = \frac{11}{3}$ $x = 1 \Rightarrow B = -\frac{1}{3}$ $\int \frac{4x^3-5}{2x^2-x-1}dx = \int\left(2x+1+\frac{\frac{11}{3}}{2x+1}+\frac{-\frac{1}{3}}{x-1}\right)dx$ $= x^2+x+\frac{11}{6}\ln\lvert 2x+1\rvert - \frac{1}{3}\ln\lvert x-1\rvert + C$
b	$\int \frac{x^2+x-1}{x^2-x}dx = \int\left(1+\frac{2x-1}{x^2-x}\right)dx$ $= x+\ln\lvert x^2-x\rvert + C$
أتحقق من فهمي صفحة 54	
a	$\int_3^4 \frac{2x^3+x^2-2x-4}{x^2-4}dx = \int_3^4\left(2x+1+\frac{6x}{x^2-4}\right)dx$ $= \left(x^2+x+3\ln\lvert x^2-4\rvert\right)\Big\|_3^4$ $= (20+3\ln 12)-(12+3\ln 5)$ $= 8+3\ln\frac{12}{5}$

b	$\frac{3x-10}{x^2-7x+12}=\frac{3x-10}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4}$ $\Rightarrow 3x-10=A(x-4)+B(x-3)$ $x=3\Rightarrow A=1$ $x=4\Rightarrow B=2$ $\int_5^6 \frac{3x-10}{x^2-7x+12}dx=\int_5^6\left(\frac{1}{x-3}+\frac{2}{x-4}\right)dx$ $=(\ln\lvert x-3\rvert+2\ln\lvert x-4\rvert)\rvert_5^6$ $=\ln 3+2\ln 2-(\ln 2+2\ln 1)$ $=\ln 3+\ln 2=\ln 6$
	أتحقق من فهمي صفحة 57
a	$u=\tan x \Rightarrow \frac{du}{dx}=\sec^2 x\Rightarrow dx=\frac{du}{\sec^2 x}$ $\int\frac{\sec^2 x}{\tan^2 x-1}dx=\int\frac{\sec^2 x}{u^2-1}\frac{du}{\sec^2 x}=\int\frac{1}{u^2-1}du$ $\frac{1}{u^2-1}=\frac{1}{(u-1)(u+1)}=\frac{A}{u-1}+\frac{B}{u+1}$ $\Rightarrow 1=A(u+1)+B(u-1)$ $u=1\Rightarrow A=\frac{1}{2}$ $u=-1\Rightarrow B=-\frac{1}{2}$ $\int\frac{1}{u^2-1}du=\int\left(\frac{\frac{1}{2}}{u-1}+\frac{-\frac{1}{2}}{u+1}\right)du$ $=\frac{1}{2}\ln\lvert u-1\rvert-\frac{1}{2}\ln\lvert u+1\rvert+C=\frac{1}{2}\ln\left\lvert\frac{u-1}{u+1}\right\rvert+C$ $\Rightarrow\int\frac{\sec^2 x}{\tan^2 x-1}dx=\frac{1}{2}\ln\left\lvert\frac{\tan x-1}{\tan x+1}\right\rvert+C$

b	$u = e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x}$ $\int \frac{e^x}{(e^x-1)(e^x+4)}\,dx = \int \frac{e^x}{(u-1)(u+4)}\,\frac{du}{e^x}$ $= \int \frac{1}{(u-1)(u+4)}\,du$ $\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$ $\Rightarrow 1 = A(u+4) + B(u-1) +$ $u = 1 \Rightarrow A = \frac{1}{5}$ $u = -4 \Rightarrow B = -\frac{1}{5}$ $\int \frac{1}{(u-1)(u+4)}\,du = \int \left(\frac{\frac{1}{5}}{u-1} + \frac{-\frac{1}{5}}{u+4} \right) du$ $= \frac{1}{5}\ln\lvert u-1\rvert - \frac{1}{5}\ln\lvert u+4\rvert + C = \frac{1}{5}\ln\left\lvert\frac{u-1}{u+4}\right\rvert + C$ $\Rightarrow \int \frac{e^x}{(e^x-1)(e^x+4)}\,dx = \frac{1}{5}\ln\left\lvert\frac{e^x-1}{e^x+4}\right\rvert + C$
أتدرب وأحل المسائل صفحة 57	
1	$\frac{x-10}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}$ $\Rightarrow x - 10 = A(x+5) + Bx$ $x = 0 \Rightarrow A = -2$ $x = -5 \Rightarrow B = 3$ $\int \frac{x-10}{x(x+5)}dx = \int \left(\frac{-2}{x} + \frac{3}{x+5}\right) dx$ $= -2\ln\lvert x\rvert + 3\ln\lvert x+5\rvert + C$

2	$\frac{2}{1-x^2}=\frac{2}{(1-x)(1+x)}=\frac{A}{1-x}+\frac{B}{1+x}$ $\Rightarrow 2=A(1+x)+B(1-x)$ $x=1\Rightarrow A=1$ $x=-1\Rightarrow B=1$ $\int \frac{2}{1-x^2}dx=\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$ $=-\ln\|1-x\|+\ln\|1+x\|+C$ $=\ln\left\|\frac{1+x}{1-x}\right\|+C$
3	$\frac{4}{(x-2)(x-4)}=\frac{A}{x-2}+\frac{B}{x-4}$ $\Rightarrow 4=A(x-4)+B(x-2)$ $x=2\Rightarrow A=-2$ $x=4\Rightarrow B=2$ $\int \frac{4}{(x-2)(x-4)}dx=\int\left(\frac{-2}{x-2}+\frac{2}{x-4}\right)dx$ $=-2\ln\|x-2\|+2\ln\|x-4\|+C$ $=2\ln\left\|\frac{x-4}{x-2}\right\|+C$
4	$\frac{3x+4}{x^2+x}=\frac{3x+4}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$ $\Rightarrow 3x+4=A(x+1)+Bx$ $x=0\Rightarrow A=4$ $x=-1\Rightarrow B=-1$ $\int \frac{3x+4}{x^2+x}dx=\int\left(\frac{4}{x}+\frac{-1}{x+1}\right)dx$ $=4\ln\|x\|-\ln\|x+1\|+C$

5	$\int \frac{x^2}{x^2-4}dx = \int \left(1+\frac{4}{x^2-4}\right)dx$ $\frac{4}{x^2-4} = \frac{4}{(x-2)(x+2)} = \frac{A}{x-2}+\frac{B}{x+2}$ $\Rightarrow 4 = A(x+2)+B(x-2)$ $x = 2 \Rightarrow A = 1$ $x = -2 \Rightarrow B = -1$ $\int \frac{x^2}{x^2-4}dx = \int \left(1+\frac{1}{x-2}+\frac{-1}{x+2}\right)dx$ $= x+\ln\lvert x-2\rvert-\ln\lvert x+2\rvert+C$ $= x+\ln\left\lvert\frac{x-2}{x+2}\right\rvert+C$
6	$\frac{3x-6}{x^2+x-2} = \frac{3x-6}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$ $\Rightarrow 3x-6 = A(x-1)+B(x+2)$ $x = -2 \Rightarrow A = 4$ $x = 1 \Rightarrow B = -1$ $\int \frac{3x-6}{x^2+x-2}dx = \int \left(\frac{4}{x+2}+\frac{-1}{x-1}\right)dx$ $= 4\ln\lvert x+2\rvert-\ln\lvert x-1\rvert+C$
7	$\frac{4x+10}{4x^2-4x-3} = \frac{4x+10}{(2x-3)(2x+1)} = \frac{A}{2x-3}+\frac{B}{2x+1}$ $\Rightarrow 4x+10 = A(2x+1)+B(2x-3)$ $x = \frac{3}{2} \Rightarrow A = 4$ $x = -\frac{1}{2} \Rightarrow B = -2$ $\int \frac{4x+10}{4x^2-4x-3}dx = \int \left(\frac{4}{2x-3}+\frac{-2}{2x+1}\right)dx$ $= 2\ln\lvert 2x-3\rvert-\ln\lvert 2x+1\rvert+C$

8	$\dfrac{2x^2+9x-11}{x^3+2x^2-5x-6}=\dfrac{2x^2+9x-11}{(x-2)(x+1)(x+3)}=\dfrac{A}{x-2}+\dfrac{B}{x+1}+\dfrac{C}{x+3}$ $\Rightarrow 2x^2+9x-11=A(x+1)(x+3)+B(x-2)(x+3)+C(x-2)(x+1)$ $x=2\Rightarrow A=1$ $x=-1\Rightarrow B=3$ $x=-3\Rightarrow C=-2$ $\displaystyle\int\frac{2x^2+9x-11}{x^3+2x^2-5x-6}dx=\int\left(\frac{1}{x-2}+\frac{3}{x+1}+\frac{-2}{x+3}\right)dx$ $=\ln\lvert x-2\rvert+3\ln\lvert x+1\rvert-2\ln\lvert x+3\rvert+C$
9	$\dfrac{4x}{x^2-2x-3}=\dfrac{4x}{(x-3)(x+1)}=\dfrac{A}{x-3}+\dfrac{B}{x+1}$ $\Rightarrow 4x=A(x+1)+B(x-3)$ $x=3\Rightarrow A=3$ $x=-1\Rightarrow B=1$ $\displaystyle\int\frac{4x}{x^2-2x-3}dx=\int\left(\frac{3}{x-3}+\frac{1}{x+1}\right)dx$ $=3\ln\lvert x-3\rvert+\ln\lvert x+1\rvert+C$
10	$\dfrac{8x^2-19x+1}{(2x+1)(x-2)^2}=\dfrac{A}{2x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$ $\Rightarrow 8x^2-19x+1=A(x-2)^2+B(2x+1)(x-2)+C(2x+1)$ $x=-\dfrac{1}{2}\Rightarrow A=2$ $x=2\Rightarrow C=-1$ $x=0\Rightarrow 1=4A-2B+C\Rightarrow B=3$ $\displaystyle\int\frac{8x^2-19x+1}{(2x+1)(x-2)^2}dx=\int\left(\frac{2}{2x+1}+\frac{3}{x-2}+\frac{-1}{(x-2)^2}\right)dx$ $=\ln\lvert 2x+1\rvert+3\ln\lvert x-2\rvert+\dfrac{1}{x-2}+C$

11	$\int \frac{9x^2-3x+2}{9x^2-4}dx = \int \left(1+\frac{6-3x}{9x^2-4}\right)dx$ $\frac{6-3x}{9x^2-4} = \frac{6-3x}{(3x-2)(3x+2)} = \frac{A}{3x-2}+\frac{B}{3x+2}$ $\Rightarrow 6-3x = A(3x+2)+B(3x-2)$ $x=\frac{2}{3} \Rightarrow A=1$ $x=-\frac{2}{3} \Rightarrow B=-2$ $\int \frac{9x^2-3x+2}{9x^2-4}dx = \int \left(1+\frac{1}{3x-2}+\frac{-2}{3x+2}\right)dx$ $= x+\frac{1}{3}\ln\|3x-2\|-\frac{2}{3}\ln\|3x+2\|+C$
12	$\int \frac{x^3+2x^2+2}{x^2+x}dx = \int \left(x+1+\frac{2-x}{x^2+x}\right)dx$ $\frac{2-x}{x^2+x} = \frac{2-x}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$ $\Rightarrow 2-x = A(x+1)+Bx$ $x=0 \Rightarrow A=2$ $x=-1 \Rightarrow B=-3$ $\int \frac{x^3+2x^2+2}{x^2+x}dx = \int \left(x+1+\frac{2}{x}+\frac{-3}{x+1}\right)dx$ $= \frac{1}{2}x^2+x+2\ln\|x\|-3\ln\|x+1\|+C$

13	$\int \frac{x^2+x+2}{3-2x-x^2}dx = \int \left(-1+\frac{5-x}{-x^2-2x+3}\right)dx$ $\frac{5-x}{-x^2-2x+3} = \frac{x-5}{(x+3)(x-1)} = \frac{A}{x+3}+\frac{B}{x-1}$ $\Rightarrow x-5 = A(x-1)+B(x+3)$ $x=-3 \Rightarrow A=2$ $x=1 \Rightarrow B=-1$ $\int \frac{x^2+x+2}{3-2x-x^2}dx = \int \left(-1+\frac{2}{x+3}+\frac{-1}{x-1}\right)dx$ $= -x+2\ln\|x+3\|-\ln\|x-1\|+C$
14	$\frac{2x-4}{(x^2+4)(x+2)} = \frac{A}{x+2}+\frac{Bx+C}{x^2+4}$ $\Rightarrow 2x-4 = A(x^2+4)+(Bx+C)(x+2)$ $x=-2 \Rightarrow A=-1$ $x=0 \Rightarrow -4=4A+2C \Rightarrow C=0$ $x=1 \Rightarrow -2=5A+3B+3C \Rightarrow B=1$ $\int \frac{2x-4}{(x^2+4)(x+2)}dx = \int \left(\frac{-1}{x+2}+\frac{x}{x^2+4}\right)dx$ $= -\ln\|x+2\|+\frac{1}{2}\ln\|x^2+4\|+C$

15	$\int \frac{x^3-4x^2-2}{x^3+x^2}\,dx = \int \left(1+\frac{-5x^2-2}{x^3+x^2}\right) dx$ $\frac{-5x^2-2}{x^3+x^2} = \frac{-5x^2-2}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$ $\Rightarrow -5x^2-2 = Ax(x+1)+B(x+1)+Cx^2$ $x=0 \Rightarrow B=-2$ $x=-1 \Rightarrow C=-7$ $x=1 \Rightarrow -7=2A+2B+C \Rightarrow A=2$ $\int \frac{x^3-4x^2-2}{x^3+x^2}\,dx = \int \left(1+\frac{2}{x}+\frac{-2}{x^2}+\frac{-7}{x+1}\right) dx$ $= x+2\ln\lvert x\rvert+\frac{2}{x}-7\ln\lvert x+1\rvert+C$
16	$\frac{3-x}{2-5x-12x^2} = \frac{x-3}{12x^2+5x-2} = \frac{x-3}{(4x-1)(3x+2)} = \frac{A}{4x-1}+\frac{B}{3x+2}$ $\Rightarrow x-3 = A(3x+2)+B(4x-1)$ $x=\frac{1}{4} \Rightarrow A=-1$ $x=-\frac{2}{3} \Rightarrow B=1$ $\int \frac{3-x}{2-5x-12x^2}\,dx = \int \left(\frac{-1}{4x-1}+\frac{1}{3x+2}\right) dx$ $= -\frac{1}{4}\ln\lvert 4x-1\rvert+\frac{1}{3}\ln\lvert 3x+2\rvert+C$

17	$\frac{3x^3-x^2+12x-6}{x^4+6x^2}=\frac{3x^3-x^2+12x-6}{x^2(x^2+6)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+6}$ $\Rightarrow 3x^3-x^2+12x-6=Ax(x^2+6)+B(x^2+6)+(Cx+D)(x^2)$ $x=0\Rightarrow B=-1$ $x=1\Rightarrow 8=7A+7B+C+D$.................(1) $x=-1\Rightarrow -22=-7A+7B-C+D$.......(2) $x=2\Rightarrow 38=20A+10B+8C+4D$...... (3) بجمع(1) ، و (2) ينتج أنّ: $14B+2D=-14$ ، وبتعويض $B=-1$، نجد أن $D=0$ وبطرح (2) من (1) ينتج أنّ: $14A+2C=30$ أي أنّ $C=15-7A$ وبالتعويض في (3) ينتج أنّ: $20A-10+8(15-7A)=38$ $-36A=-72\Rightarrow A=2$ $C=15-7(2)=1$ $\int\frac{3x^3-x^2+12x-6}{x^4+6x^2}dx=\int\left(\frac{2}{x}+\frac{-1}{x^2}+\frac{x}{x^2+6}\right)dx$ $=2\ln\|x\|+\frac{1}{x}+\frac{1}{2}\ln\|x^2+6\|+C$
18	$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$ $\Rightarrow 5x-2=A(x-2)+B$ $x=2\Rightarrow B=8$ $x=0\Rightarrow -2=-2A+B\Rightarrow A=5$ $\int\frac{5x-2}{(x-2)^2}dx=\int\left(\frac{5}{x-2}+\frac{8}{(x-2)^2}\right)dx$ $=5\ln\|x-2\|-\frac{8}{x-2}+C$ ملاحظة: يمكن حل هذا التكامل بالتعويض $u=x-2$، كما يمكن حله بالأجزاء حيث: $u=5x-2, dv=(x-2)^{-2}$

19	$\frac{6+3x-x^2}{x^3+2x^2} = \frac{6+3x-x^2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$ $\Rightarrow 6+3x-x^2 = Ax(x+2) + B(x+2) + C(x^2)$ $x = 0 \Rightarrow B = 3$ $x = -2 \Rightarrow C = -1$ $x = 1 \Rightarrow 8 = 3A + 3B + C \Rightarrow A = 0$ $\int_2^4 \frac{6+3x-x^2}{x^3+2x^2}\, dx = \int_2^4 \left(\frac{3}{x^2} + \frac{-1}{x+2}\right) dx$ $= \left(-\frac{3}{x} - \ln\lvert x+2\rvert\right)\Big\rvert_2^4$ $= -\frac{3}{4} - \ln 6 + \frac{3}{2} + \ln 4 = \frac{3}{4} + \ln\frac{2}{3}$
20	$\frac{9x^2+4}{9x^2-4} = 1 + \frac{8}{9x^2-4}$ $\frac{8}{9x^2-4} = \frac{8}{(3x-2)(3x+2)} = \frac{A}{3x-2} + \frac{B}{3x+2}$ $\Rightarrow 8 = A(3x+2) + B(3x-2)$ $x = \frac{2}{3} \Rightarrow A = 2$ $x = -\frac{2}{3} \Rightarrow B = -2$ $\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{9x^2+4}{9x^2-4}\, dx = \int_{-\frac{1}{3}}^{\frac{1}{3}} \left(1 + \frac{2}{3x-2} + \frac{-2}{3x+2}\right) dx$ $= \left(x + \frac{2}{3}\ln\lvert 3x-2\rvert - \frac{2}{3}\ln\lvert 3x+2\rvert\right)\Big\rvert_{-\frac{1}{3}}^{\frac{1}{3}}$ $= \left(x + \frac{2}{3}\ln\left\lvert\frac{3x-2}{3x+2}\right\rvert\right)\Big\rvert_{-\frac{1}{3}}^{\frac{1}{3}}$ $= \frac{1}{3} + \frac{2}{3}\ln\frac{1}{3} + \frac{1}{3} - \frac{2}{3}\ln 3 = \frac{2}{3} - \frac{4}{3}\ln 3$

21	$\frac{17-5x}{(2x+3)(2-x)^2}=\frac{A}{2x+3}+\frac{B}{2-x}+\frac{C}{(2-x)^2}$ $\Rightarrow 17-5x=A(2-x)^2+B(2-x)(2x+3)+C(2x+3)$ $x=-\frac{3}{2}\Rightarrow A=2$ $x=2\Rightarrow C=1$ $x=0\Rightarrow 17=4A+6B+3C\Rightarrow B=1$ $\int_0^1 \frac{17-5x}{(2x+3)(2-x)^2}\,dx=\int_0^1\left(\frac{2}{2x+3}+\frac{1}{2-x}+\frac{1}{(2-x)^2}\right)dx$ $=\left(\ln\lvert 2x+3\rvert-\ln\lvert 2-x\rvert+\frac{1}{2-x}\right)\Big\rvert_0^1$ $=\ln 5+1-\ln 3+\ln 2-\frac{1}{2}=\frac{1}{2}+\ln\frac{10}{3}$
22	$\frac{4}{16x^2+8x-3}=\frac{4}{(4x-1)(4x+3)}=\frac{A}{4x-1}+\frac{B}{4x+3}$ $\Rightarrow 4=A(4x+3)+B(4x-1)$ $x=\frac{1}{4}\Rightarrow A=1$ $x=-\frac{3}{4}\Rightarrow B=-1$ $\int_1^4 \frac{4}{16x^2+8x-3}\,dx=\int_1^4\left(\frac{1}{4x-1}+\frac{-1}{4x+3}\right)dx$ $=\left(\frac{1}{4}\ln\lvert 4x-1\rvert-\frac{1}{4}\ln\lvert 4x+3\rvert\right)\Big\rvert_1^4$ $=\left(\frac{1}{4}\ln\left\lvert\frac{4x-1}{4x+3}\right\rvert\right)\Big\rvert_1^4$ $=\frac{1}{4}\left(\ln\frac{15}{19}-\ln\frac{3}{7}\right)=\frac{1}{4}\ln\frac{35}{19}$

23	$\frac{5x+5}{x^2+x-6}=\frac{5x+5}{(x-2)(x+3)}=\frac{A}{x-2}+\frac{B}{x+3}$ $\Rightarrow 5x+5=A(x+3)+B(x-2)$ $x=2\Rightarrow A=3$ $x=-3\Rightarrow B=2$ $\int_3^4\frac{5x+5}{x^2+x-6}\,dx=\int_3^4\left(\frac{3}{x-2}+\frac{2}{x+3}\right)dx$ $=(3\ln\|x-2\|+2\ln\|x+3\|)\|_3^4$ $=3\ln 2+2\ln 7-2\ln 6=\ln\frac{98}{9}$
24	$\frac{4}{x^3-4x^2+4x}=\frac{4}{x(x-2)^2}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$ $\Rightarrow 4=A(x-2)^2+Bx(x-2)+Cx$ $x=0\Rightarrow A=1$ $x=2\Rightarrow C=2$ $x=1\Rightarrow 4=A-B+C\Rightarrow B=-1$ $A=\int_3^4\frac{4}{x^3-4x^2+4x}\,dx=\int_3^4\left(\frac{1}{x}+\frac{-1}{x-2}+\frac{2}{(x-2)^2}\right)dx$ $=\left(\ln\|x\|-\ln\|x-2\|-\frac{2}{x-2}\right)\Big\|_3^4$ $=\left(\ln\left\|\frac{x}{x-2}\right\|-\frac{2}{x-2}\right)\Big\|_3^4$ $=\ln 2-1-\ln 3+2=1+\ln\frac{2}{3}$

25	$A = \int_0^1 \frac{1}{x^2 - 5x + 6}\, dx$ $\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$ $\Rightarrow 1 = A(x-2) + B(x-3)$ $x = 3 \Rightarrow A = 1$ $x = 2 \Rightarrow B = -1$ $A = \int_0^1 \frac{1}{x^2 - 5x + 6}\, dx = \int_0^1 \left(\frac{1}{x-3} + \frac{-1}{x-2}\right) dx$ $= (\ln\lvert x-3\rvert - \ln\lvert x-2\rvert)\big\vert_0^1$ $= \ln\left\lvert\frac{x-3}{x-2}\right\rvert\Big\vert_0^1$ $= \ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3}$

26	$A=\int_1^2 \frac{x^2+1}{3x-x^2}\,dx$ $\frac{x^2+1}{3x-x^2}=-1+\frac{3x+1}{3x-x^2}$ $\frac{3x+1}{3x-x^2}=\frac{3x+1}{x(3-x)}=\frac{A}{x}+\frac{B}{3-x}$ $\Rightarrow 3x+1=A(3-x)+Bx$ $x=0\Rightarrow A=\frac{1}{3}$ $x=3\Rightarrow B=\frac{10}{3}$ $A=\int_1^2 \frac{x^2+1}{3x-x^2}\,dx=\int_1^2\left(-1+\frac{\frac{1}{3}}{x}+\frac{\frac{10}{3}}{3-x}\right)dx$ $=\left(-x+\frac{1}{3}\ln\|x\|-\frac{10}{3}\ln\|3-x\|\right)\Big\|_1^2$ $=-2+\frac{1}{3}\ln 2+1+\frac{10}{3}\ln 2$ $=-1+\frac{11}{3}\ln 2$
27	$f(x)=0 \Rightarrow 4x-5=0 \Rightarrow x=\frac{5}{4} \Rightarrow A\left(\frac{5}{4},0\right)$
28	$A=\int_0^{\frac{5}{4}} \frac{4x-5}{2x^2-5x-3}\,dx=\ln\|2x^2-5x-3\|\Big\|_0^{\frac{5}{4}}=\ln\frac{49}{8}-\ln 3=\ln\frac{49}{24}$ ملاحظة: البسط هو مشتقة المقام، فلا داعي لتجزئة الكسر.

29	$u=\cos x \Rightarrow \frac{du}{dx}=-\sin x \Rightarrow dx=\frac{du}{-\sin x}$ $\int \frac{\sin x}{\cos x+\cos^2 x}\,dx=\int \frac{\sin x}{u+u^2}\times\frac{du}{-\sin x}=\int \frac{-1}{u+u^2}\,du$ $\frac{-1}{u+u^2}=\frac{-1}{u(1+u)}=\frac{A}{u}+\frac{B}{1+u}$ $\Rightarrow -1=A(1+u)+Bu$ $u=0 \Rightarrow A=-1$ $u=-1 \Rightarrow B=1$ $\int \frac{-1}{u+u^2}\,du=\int\left(\frac{-1}{u}+\frac{1}{1+u}\right)du$ $=-\ln\lvert u\rvert+\ln\lvert 1+u\rvert+C$ $\Rightarrow \int \frac{\sin x}{\cos x+cos^2 x}\,dx=-\ln\lvert\cos x\rvert+\ln\lvert 1+\cos x\rvert+C$ $=\ln\left\lvert\frac{1+\cos x}{\cos x}\right\rvert+C=\ln\lvert 1+\sec x\rvert+C$

30	$u=\sqrt{x} \Rightarrow u^2=x \Rightarrow dx=2udu$ $\int \frac{1}{x^2+x\sqrt{x}}\,dx=\int \frac{1}{u^4+u^3}\,2udu=\int \frac{2}{u^3+u^2}\,du$ $\frac{2}{u^3+u^2}=\frac{2}{u^2(u+1)}=\frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1}$ $\Rightarrow 2=Au(u+1)+B(u+1)+Cu^2$ $u=0 \Rightarrow B=2$ $u=-1 \Rightarrow C=2$ $u=1 \Rightarrow 2=2A+2B+C \Rightarrow A=-2$ $\int \frac{2}{u^3+u^2}\,du=\int\left(\frac{-2}{u}+\frac{2}{u^2}+\frac{2}{u+1}\right)du$ $=-2\ln\lvert u\rvert-\frac{2}{u}+2\ln\lvert u+1\rvert+C$ $=2\ln\left\lvert\frac{u+1}{u}\right\rvert-\frac{2}{u}+C$ $\Rightarrow \int \frac{1}{x^2+x\sqrt{x}}\,dx=2\ln\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)-\frac{2}{\sqrt{x}}+C$

31	$u = e^x \Rightarrow \frac{du}{dx} = e^x = u \Rightarrow dx = \frac{du}{u}$ $\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\, dx = \int \frac{u^2}{u^2 + 3u + 2} \times \frac{du}{u} = \int \frac{u}{u^2 + 3u + 2}\, du$ $\frac{u}{u^2 + 3u + 2} = \frac{u}{(u + 1)(u + 2)} = \frac{A}{u + 1} + \frac{B}{u + 2}$ $\Rightarrow u = A(u + 2) + B(u + 1)$ $u = -1 \Rightarrow A = -1$ $u = -2 \Rightarrow B = 2$ $\int \frac{u}{u^2 + 3u + 2}\, du = \int \left(\frac{-1}{u + 1} + \frac{2}{u + 2}\right) du$ $= -\ln\|u + 1\| + 2\ln\|u + 2\| + C$ $\Rightarrow \int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\, dx = -\ln(e^x + 1) + 2\ln(e^x + 2) + C$

32	$u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$ $\int \frac{\cos x}{\sin x\,(\sin^2 x - 4)}\,dx = \int \frac{\cos x}{u(u^2-4)} \times \frac{du}{\cos x} = \int \frac{1}{u(u^2-4)}\,du$ $\frac{1}{u(u^2-4)} = \frac{1}{u(u-2)(u+2)} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{u+2}$ $\Rightarrow 1 = A(u-2)(u+2) + Bu(u+2) + Cu(u-2)$ $u = 0 \Rightarrow A = -\frac{1}{4}$ $u = 2 \Rightarrow B = \frac{1}{8}$ $u = -2 \Rightarrow C = \frac{1}{8}$ $\int \frac{1}{u(u^2-4)}\,du = \int \left(\frac{-\frac{1}{4}}{u} + \frac{\frac{1}{8}}{u-2} + \frac{\frac{1}{8}}{u+2} \right) du$ $= -\frac{1}{4}\ln\|u\| + \frac{1}{8}\ln\|u-2\| + \frac{1}{8}\ln\|u+2\| + C$ $\Rightarrow \int \frac{\cos x}{\sin x\,(sin^2 x - 4)}\,dx$ $= -\frac{1}{4}\ln\|\sin x\| + \frac{1}{8}\ln\|\sin x - 2\| + \frac{1}{8}\ln\|\sin x + 2\| + C$

33	الحل الأول بضرب كل من البسط والمقام بـ e^{-x} $\int \frac{1}{1+e^x}dx = \int \frac{e^{-x}}{e^{-x}+1}dx = -\int \frac{-e^{-x}}{e^{-x}+1}dx = -\ln(e^{-x}+1)+C$ الحل الثاني بالتعويض: $u = e^x \Rightarrow \frac{du}{dx} = e^x = u \Rightarrow dx = \frac{du}{u}$ $\int \frac{1}{1+e^x}dx = \int \frac{1}{1+u} \times \frac{du}{u} = \int \frac{1}{u(1+u)}du$ $\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{u+1}$ $\Rightarrow 1 = A(1+u) + Bu$ $u = 0 \Rightarrow A = 1$ $u = -1 \Rightarrow B = -1$ $\int \frac{1}{u(1+u)}du = \int \left(\frac{1}{u} + \frac{-1}{u+1}\right)du = \ln\lvert u\rvert - \ln\lvert u+1\rvert + C$ $\Rightarrow \int \frac{1}{1+e^x}dx = \ln e^x - \ln(e^x+1) + C$ $= \ln\left(\frac{e^x+1}{e^x}\right)^{-1} + C = -\ln(e^{-x}+1) + C$
34	$\int_0^{\ln 2} \frac{1}{1+e^x}dx = \ln e^x - \ln(e^x+1)\vert_0^{\ln 2}$ $= \ln e^{\ln 2} - \ln(e^{\ln 2}+1) - (\ln e^0 - \ln(e^0+1))$ $= \ln 2 - \ln 3 - 0 + \ln 2 = \ln 4 - \ln 3 = \ln\frac{4}{3}$

35	$\frac{5x^2-8x+1}{2x(x-1)^2}=\frac{A}{2x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ $\Rightarrow 5x^2-8x+1=A(x-1)^2+B(2x)(x-1)+C(2x)$ $x=0\Rightarrow A=1$ $x=1\Rightarrow C=-1$ $x=-1\Rightarrow 14=4A+4B-2C\Rightarrow B=2$ $\int_4^9\frac{5x^2-8x+1}{2x(x-1)^2}\,dx=\int_4^9\left(\frac{1}{2x}+\frac{2}{x-1}+\frac{-1}{(x-1)^2}\right)dx$ $=\left(\frac{1}{2}\ln\lvert x\rvert+2\ln\lvert x-1\rvert+\frac{1}{x-1}\right)\Big\rvert_4^9$ $=\frac{1}{2}\ln 9+2\ln 8+\frac{1}{8}-\frac{1}{2}\ln 4-2\ln 3-\frac{1}{3}$ $=\ln 3+\ln 64+\frac{1}{8}-\ln 2-\ln 9-\frac{1}{3}$ $=\ln\frac{3(64)}{2(9)}-\frac{5}{24}=\ln\frac{32}{3}-\frac{5}{24}$

36	$u=\sqrt{x} \Rightarrow u^2 = x \Rightarrow dx = 2udu$ $x = 9 \Rightarrow u = 3$ $x = 16 \Rightarrow u = 4$ $\int_9^{16} \frac{2\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{2u}{u^2-4}\,2udu = \int_3^4 \frac{4u^2}{u^2-4}\,du$ $= \int_3^4 \left(4 + \frac{16}{u^2-4}\right) du$ $\frac{16}{u^2-4} = \frac{16}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$ $\Rightarrow 16 = A(u+2) + B(u-2)$ $u = 2 \Rightarrow A = 4$ $u = -2 \Rightarrow B = -4$ $\int_3^4 \left(4 + \frac{16}{u^2-4}\right) du = \int_3^4 \left(4 + \frac{4}{u-2} + \frac{-4}{u+2}\right) du$ $= (4u + 4\ln\|u-2\| - 4\ln\|u+2\|)\|_3^4$ $= 16 + 4\ln 2 - 4\ln 6 - 12 - 4\ln 1 + 4\ln 5$ $= 4 + 4\ln\frac{5}{3} = 4(1 + \ln\frac{5}{3})$ $\Rightarrow \int_9^{16} \frac{2\sqrt{x}}{x-4}\,dx = 4\left(1 + \ln\frac{5}{3}\right)$

	$\frac{4x^2+9x+4}{2x^2+5x+3} = 2 - \frac{x+2}{2x^2+5x+3}$ $\frac{x+2}{2x^2+5x+3} = \frac{x+2}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3}$ $\Rightarrow x+2 = A(2x+3) + B(x+1)$ $x = -1 \Rightarrow A = 1$ $x = -\frac{3}{2} \Rightarrow B = -1$
37	$\int_0^1 \frac{4x^2+9x+4}{2x^2+5x+3}\, dx = \int_0^1 \left(2 - \frac{1}{x+1} + \frac{1}{2x+3}\right) dx$ $= \left(2x - \ln\|x+1\| + \frac{1}{2}\ln\|2x+3\|\right)\Big\|_0^1$ $= 2 - \ln 2 + \frac{1}{2}\ln 5 - 0 + \ln 1 - \frac{1}{2}\ln 3$ $= 2 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 5 - \frac{1}{2}\ln 3$ $= 2 + \frac{1}{2}(\ln 5 - \ln 4 - \ln 3) = 2 + \frac{1}{2}\ln\frac{5}{12}$

38	$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ $u=\sqrt{1+\sqrt{x}} \Rightarrow \frac{du}{dx}=\frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}},\quad 1+\sqrt{x}=u^2 \Rightarrow \sqrt{x}=u^2-1$ $\Rightarrow dx=4\sqrt{x}\sqrt{1+\sqrt{x}}du=4u(u^2-1)du$ $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx=\int \frac{u}{(u^2-1)^2}4u(u^2-1)du=\int \frac{4u^2}{u^2-1}\,du$ $\frac{4u^2}{u^2-1}=4+\frac{4}{u^2-1}$ $\frac{4}{u^2-1}=\frac{4}{(u-1)(u+1)}=\frac{A}{u-1}+\frac{B}{u+1}$ $\Rightarrow 4=A(u+1)+B(u-1)$ $u=1 \Rightarrow A=2$ $u=-1 \Rightarrow B=-2$ $\int \frac{4u^2}{u^2-1}\,du=\int\left(4+\frac{2}{u-1}+\frac{-2}{u+1}\right)du$ $=4u+2\ln\lvert u-1\rvert-2\ln\lvert u+1\rvert+C$ $=4u+2\ln\left\lvert\frac{u-1}{u+1}\right\rvert+C$ $\Rightarrow \int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx=4\sqrt{1+\sqrt{x}}+2\ln\left\lvert\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right\rvert+C$

39	$\frac{x}{16x^4-1}=\frac{x}{(4x^2+1)(2x-1)(2x+1)}=\frac{Ax+B}{4x^2+1}+\frac{C}{2x-1}+\frac{D}{2x+1}$ $\Rightarrow x=(Ax+B)(2x-1)(2x+1)+C(4x^2+1)(2x+1)+D(4x^2+1)(2x-1)$ $x=\frac{1}{2}\Rightarrow C=\frac{1}{8}$ $x=-\frac{1}{2}\Rightarrow D=\frac{1}{8}$ $x=0\Rightarrow 0=-B+C-D\Rightarrow B=0$ $x=1\Rightarrow 1=3A+3B+15C+5D\Rightarrow A=-\frac{1}{2}$ $\int\frac{x}{16x^4-1}\,dx=\int\left(\frac{-\frac{1}{2}x}{4x^2+1}+\frac{\frac{1}{8}}{2x-1}+\frac{\frac{1}{8}}{2x+1}\right)dx$ $=-\frac{1}{16}\ln(4x^2+1)+\frac{1}{16}\ln\|2x-1\|+\frac{1}{16}\ln\|2x+1\|+C$ $=\frac{1}{16}\ln\left\|\frac{4x^2-1}{4x^2+1}\right\|+C$
40	$u=x^{\frac{1}{6}}\Rightarrow\quad\frac{du}{dx}\Rightarrow\frac{1}{6}x^{\frac{-5}{6}}\Rightarrow dx=6x^{\frac{5}{6}}du=6u^5du$ $u=x^{\frac{1}{6}}\Rightarrow x=u^6\Rightarrow\sqrt{x}=u^3,\sqrt[3]{x}=u^2$ $\Rightarrow\int\frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx=\int\frac{6u^5}{u^3-u^2}\,du$ $=\int\frac{6u^3}{u-1}\,du$ $=\int\left(6u^2+6u+6+\frac{6}{u-1}\right)du$ $=2u^3+3u^2+6u+6\ln\|u-1\|+C$ $=2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left\|\sqrt[6]{x}-1\right\|+C$