الدرس الثاني: مشتقتا الضرب والقسمة والمشتقات العليا

مسألة اليوم صفحة 28

$$A(b) = \frac{40 + 24b^{0.4}}{1 + 4b^{0.4}}$$

$$A'(b) = \frac{(1 + 4b^{0.4})(9.6b^{-0.6}) - (40 + 24b^{0.4})(1.6b^{-0.6})}{(1 + 4b^{0.4})^2}$$

$$= \frac{9.6b^{-0.6} + 38.4b^{-0.2} - 64b^{-0.6} - 38.4b^{-0.2}}{(1 + 4b^{0.4})^2}$$

$$= \frac{-54.4b^{-0.6}}{(1 + 4b^{0.4})^2}$$

أتحقق من فهمي صفحة 30

a

$$f(x) = (x^3 - 2x^2 + 3)(7x^2 - 4x)$$

$$f'(x) = (x^3 - 2x^2 + 3)(14x - 4) + (7x^2 - 4x)(3x^2 - 4x)$$

$$= 14x^4 - 4x^3 - 28x^3 + 8x^2 + 42x - 12 + 21x^4 - 28x^3 - 12x^3 + 16x^2$$

$$= 35x^4 - 72x^3 + 24x^2 + 42x - 12$$

b

$$f(x) = \ln x \cos x$$

$$f'(x) = (\ln x)(-\sin x) + (\cos x)\left(\frac{1}{x}\right) = -\ln x \sin x + \frac{\cos x}{x}$$

أتحقق من فهمي صفحة 32

a

$$f(x) = \frac{x + 1}{2x + 1}$$

$$f'(x) = \frac{(2x + 1)(1) - (x + 1)(2)}{(2x + 1)^2} = \frac{-1}{(2x + 1)^2}$$

b

$$f(x) = \frac{\sin x}{e^x}$$

$$f'(x) = \frac{e^x(\cos x) - (\sin x)e^x}{e^{2x}} = \frac{\cos x - \sin x}{e^x}$$

أتحقق من فهمي صفحة 34

a	$P(t)=\dfrac{500t^2}{2t+9}$ $P'(t)=\dfrac{(2t+9)(1000t)-(500t^2)(2)}{(2t+9)^2}=\dfrac{9000t+1000t^2}{(2t+9)^2}$
b	$P'(12)=\dfrac{9000(12)+1000(12)^2}{(24+9)^2}\approx 231.405$ إذن في السنة 12 يتزايد عدد سكان هذه المدينة بمعدل 231 ألف نسمة سنويًا تقريبًا
أتحقق من فهمي صفحة 35	
a	$f(x)=\dfrac{1}{5x-x^2}$ $f'(x)=\dfrac{-(5-2x)}{(5x-x^2)^2}=\dfrac{2x-5}{(5x-x^2)^2}$
b	$f(x)=\dfrac{1}{e^x+\sqrt{x}}$ $f'(x)=\dfrac{-\left(e^x+\dfrac{1}{2\sqrt{x}}\right)}{\left(e^x+\sqrt{x}\right)^2}=-\dfrac{2\sqrt{x}e^x+1}{2\sqrt{x}\left(e^x+\sqrt{x}\right)^2}$
أتحقق من فهمي صفحة 37	
a	$f(x)=x\cot x$ $f'(x)=(x)(-\csc^2 x)+(\cot x)(1)=-x\csc^2 x+\cot x$
b	$f(x)=\dfrac{\tan x}{1+\sin x}$ $f'(x)=\dfrac{(1+\sin x)(\sec^2 x)-(\tan x)(\cos x)}{(1+\sin x)^2}$ $=\dfrac{\sec^2 x+\sin x\sec^2 x-\sin x}{(1+\sin x)^2}$
أتحقق من فهمي صفحة 38	

	$f'(x)=\frac{(x)(\cos x)-(\sin x)(1)}{x^2}=\frac{x\cos x-\sin x}{x^2}=\frac{\cos x}{x}-\frac{\sin x}{x^2}$ $f''(x)=\frac{(x)(-\sin x)-(\cos x)(1)}{x^2}-\frac{(x^2)(\cos x)-(\sin x)(2x)}{x^4}$ $=\frac{-x\sin x-\cos x}{x^2}-\frac{x^2\cos x-2x\sin x}{x^4}$ $=\frac{-x^2\sin x-2x\cos x+2\sin x}{x^3}$ $f'''(x)=\frac{-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x}{x^4}$ ويمكن التوصل إلى الإجابة نفسها بتحويل الاقتران إلى $f(x)=x^{-1}\sin x$ وتطبيق قاعدة مشتقة ضرب اقترانين.
أتدرب وأحل المسائل صفحة 38	
1	$f(x)=\frac{x^3}{2x-1}$ $f'(x)=\frac{(2x-1)(3x^2)-(x^3)(2)}{(2x-1)^2}=\frac{4x^3-3x^2}{(2x-1)^2}$
2	$f(x)=x^3\sec x$ $f'(x)=(x^3)(\sec x\tan x)+(\sec x)(3x^2)$ $=x^3\sec x\tan x+3x^2\sec x$
3	$f(x)=\frac{x+1}{\cos x}$ $f'(x)=\frac{(\cos x)(1)-(x+1)(-\sin x)}{(\cos x)^2}=\frac{\cos x+x\sin x+\sin x}{cos^2 x}$
4	$f(x)=e^x(\tan x-x)$ $f'(x)=(e^x)(\sec^2 x-1)+(\tan x-x)(e^x)$ $=e^x\tan^2 x+e^x\tan x-xe^x$

5	$f(x)=\dfrac{\sin x+\cos x}{e^x}$ $f'(x)=\dfrac{(e^x)(\cos x-\sin x)-(\sin x+\cos x)(e^x)}{(e^x)^2}=\dfrac{-2\sin x}{e^x}$
6	$f(x)=x^3\sin x+x^2\cos x$ $f'(x)=(x^3)(\cos x)+(\sin x)(3x^2)+(x^2)(-\sin x)+(\cos x)(2x)$ $=x^3\cos x+2x^2\sin x+2x\cos x$
7	$f(x)=\sqrt[3]{x}\left(\sqrt{x}+3\right)=x^{\frac{5}{6}}+3x^{\frac{1}{3}}$ $f'(x)=\dfrac{5}{6}x^{-\frac{1}{6}}+x^{-\frac{2}{3}}=\dfrac{5}{6\sqrt[6]{x}}+\dfrac{1}{\sqrt[3]{x^2}}$
8	$f(x)=\dfrac{1+\sec x}{1-\sec x}$ $f'(x)=\dfrac{(1-\sec x)(\sec x\tan x)-(1+\sec x)(-\sec x\tan x)}{(1-\sec x)^2}$ $=\dfrac{2\sec x\tan x}{(1-\sec x)^2}$
9	$f(x)=\dfrac{2-\frac{1}{x}}{x-3}=\dfrac{2x-1}{x^2-3x}$ $f'(x)=\dfrac{(x^2-3x)(2)-(2x-1)(2x-3)}{(x^2-3x)^2}=\dfrac{-2x^2+2x-3}{(x^2-3x)^2}$
10	$f(x)=(x^3-x)(x^2+2)(x^2+x+1)$ $f'(x)=(x^3-x)\left((x^2+2)(2x+1)+(x^2+x+1)(2x)\right)$ $+(x^2+2)(x^2+x+1)(3x^2-1)$ $=(x^3-x)(x^2+2)(2x+1)+(x^3-x)(x^2+x+1)(2x)$ $+(x^2+2)(x^2+x+1)(3x^2-1)$

11	$f(x) = (\csc x + \cot x)^{-1} = \dfrac{1}{\csc x + \cot x}$ $f'(x) = \dfrac{-1(-\csc x \cot x - \csc^2 x)}{(\csc x + \cot x)^2}$ $= \dfrac{\csc x \cot x + \csc^2 x}{(\csc x + \cot x)^2}$ $= \dfrac{\csc x (\cot x + \csc x)}{(\csc x + \cot x)^2}$ $= \dfrac{\csc x}{\cot x + \csc x}$
12	$(fg)'(0) = f(0)g'(0) + g(0)f'(0)$ $= 5 \times 2 - 1 \times -3 = 13$
13	$\left(\dfrac{f}{g}\right)'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{g^2(0)} = \dfrac{-1 \times -3 - 5 \times 2}{(-1)^2} = -7$
14	$(7f - 2fg)'(0) = 7f'(0) - 2(fg)'(0) = 7(-3) - 2(13) = -47$
15	$f(x) = \dfrac{x^2 - 4}{x^2 + 4}$ $f'(x) = \dfrac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \dfrac{16x}{(x^2+4)^2}$ $f''(x) = \dfrac{(x^2+4)^2(16) - (16x)(2)(x^2+4)^1(2x)}{(x^2+4)^4}$ $= \dfrac{(16)(x^2+4) - (16x)(2)(2x)}{(x^2+4)^3}$ $f''(-2) = \dfrac{(16)(8) - (-32)(2)(-4)}{(8)^3} = -\dfrac{1}{4}$

16

$$f(x)=\frac{1+x}{1+\sqrt[3]{x}}=\frac{(1+\sqrt[3]{x})(1-\sqrt[3]{x}+\sqrt[3]{x^2})}{1+\sqrt[3]{x}}=1-\sqrt[3]{x}+\sqrt[3]{x^2}$$

$$f'(x)=-\frac{1}{3}x^{-\frac{2}{3}}+\frac{2}{3}x^{-\frac{1}{3}}$$

$$f''(x)=\frac{2}{9}x^{-\frac{5}{3}}-\frac{2}{9}x^{-\frac{4}{3}}=\frac{2}{9\sqrt[3]{x^5}}-\frac{2}{9\sqrt[3]{x^4}}$$

$$f''(8)=\frac{2}{9\sqrt[3]{8^5}}-\frac{2}{9\sqrt[3]{8^4}}=\frac{2}{9}\left(\frac{1}{32}-\frac{1}{16}\right)=-\frac{1}{144}$$

17

$$f(x)=\frac{1}{1+\sqrt{x}}$$

$$f'(x)=\frac{-\left(\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2}=\frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}$$

$$f''(x)=\frac{2\sqrt{x}(2)(1+\sqrt{x})^1\left(\frac{1}{2\sqrt{x}}\right)+(1+\sqrt{x})^2\left(\frac{1}{\sqrt{x}}\right)}{4x(1+\sqrt{x})^4}$$

$$=\frac{2+\frac{1+\sqrt{x}}{\sqrt{x}}}{4x(1+\sqrt{x})^3}$$

$$f''(4)=\frac{2+\frac{1+2}{2}}{16(1+2)^3}=\frac{7}{864}$$

18

$$f(x)=\frac{1+x}{1+e^x}$$

$$f'(x)=\frac{(1+e^x)(1)-(1+x)(e^x)}{(1+e^x)^2}=\frac{1-xe^x}{(1+e^x)^2}$$

ميل المماس عند النقطة $\left(0,\frac{1}{2}\right)$ هو: $f'(0)=\frac{1}{4}$

معادلة المماس هي:

$$y-\frac{1}{2}=\frac{1}{4}(x-0)\quad\rightarrow y=\frac{1}{4}x+\frac{1}{2}$$

19	$f(x) = e^x \cos x + \sin x$ $f'(x) = (e^x)(-\sin x) + (\cos x)(e^x) + \cos x$ ميل المماس عند النقطة $(0, 1)$ هو: $f'(0) = (1)(0) + (1)(1) + 1 = 2$ معادلة المماس هي: $y - 1 = 2(x - 0) \quad \rightarrow y = 2x + 1$
20	$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ $= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$ $= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$ $= -\frac{1}{\sin^2 x}$ $= -\csc^2 x$
21	$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$ $= \frac{-(-\sin x)}{\cos^2 x}$ $= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$ $= \sec x \tan x$
22	$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$ $= \frac{-(\cos x)}{\sin^2 x}$ $= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$ $= -\csc x \cot x$

23	$f''(x) = 2 - \frac{2}{x}$ $f'''(x) = \frac{2}{x^2}$
24	$f'''(x) = 2\sqrt{x}$ $f^{(4)}(x) = \frac{1}{\sqrt{x}}$
25	$f^{(4)}(x) = 2x + 1$ $f^{(5)}(x) = 2$ $f^{(6)}(x) = 0$
26	$h(t) = \frac{3t^2}{4 + t^2}$ $h'(t) = \frac{(4 + t^2)(6t) - (3t^2)(2t)}{(4 + t^2)^2} = \frac{24t}{(4 + t^2)^2}$
27	$y = e^x \sin x$ $\frac{dy}{dx} = (e^x)(\cos x) + (\sin x)(e^x) = e^x(\cos x + \sin x)$ $\frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + e^x(\cos x + \sin x) = 2e^x \cos x$
28	$2\frac{dy}{dx} - 2y = 2e^x(\cos x + \sin x) - 2e^x \sin x$ $= 2e^x \cos x$ $= \frac{d^2y}{dx^2}$
29	$\csc\theta = \frac{r + h}{r} \rightarrow r + h = r\csc\theta$ $\rightarrow h = r(\csc\theta - 1)$

30	$\frac{dh}{d\theta} = r(-\csc\theta\cot\theta)$ $\left.\frac{dh}{d\theta}\right\|_{\theta=\frac{\pi}{6}} = 6371\left(-\csc\frac{\pi}{6}\cot\frac{\pi}{6}\right)$ $= 6371\left(-2\times\sqrt{3}\right) \approx -22070\ km/rad$
31	$f(x) = 9\ln x + \frac{1}{2x^2}$ $f'(x) = 9\left(\frac{1}{x}\right) + \frac{-1(4x)}{4x^4}$ $= \frac{9}{x} - \frac{1}{x^3}$ $= \frac{9x^2-1}{x^3}$ $= \frac{(3x-1)(3x+1)}{x^3}$
32	$P'(2) = F(2)G'(2) + G(2)F'(2)$ $G'(2)$ ميل المستقيم الذي يمر بالنقطتين $(2,2)$و$(4,3)$ ويساوي $\frac{1}{2}$ $F'(2)$ ميل الماس الأفقي، ويساوي صفرًا $P'(2) = 3\times\frac{1}{2} + 2\times 0 = \frac{3}{2}$
33	$Q'(7) = \frac{G(7)F'(7) - F(7)G'(7)}{G^2(7)} = \frac{1\times\frac{1}{4} - 5\times -\frac{2}{3}}{1} = \frac{43}{12}$

34	$y=\frac{1-e^{-x}}{1+e^{-x}}=\frac{1-\frac{1}{e^x}}{1+\frac{1}{e^x}}$ $=\frac{e^x-1}{e^x+1}$ $\frac{dy}{dx}=\frac{(e^x+1)(e^x)-(e^x-1)(e^x)}{(e^x+1)^2}=\frac{2e^x}{(e^x+1)^2}$ $\left.\frac{dy}{dx}\right\|_{x=0}=\frac{2(1)}{(1+1)^2}=\frac{1}{2}$
35	إذا وجد مماس أفقي فإن ميله يساوي صفرًا، أي أنّ : $\frac{2e^x}{(e^x+1)^2}=0$ ، وهذا لا يتحقق إلّا إذا كان $e^x=0$، ولكن $e^x>0$ لجميع الأعداد الحقيقية x، ولذا لا يوجد لهذا المنحنى مماسات أفقية.
36	$y=\frac{x+1}{x-1}$ $\frac{dy}{dx}=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}=\frac{-2}{(x-1)^2}$
37	$y=\frac{x+1}{x-1} \rightarrow x+1=y(x-1) \rightarrow x(1-y)=-y-1$ $x=\frac{y+1}{y-1}$ $\frac{dx}{dy}=\frac{-2}{(y-1)^2}$
38	$\frac{dx}{dy}=\frac{-2}{(y-1)^2}$ $=\frac{-2}{\left(\frac{x+1}{x-1}-1\right)^2}$ $=\frac{-2}{\left(\frac{2}{x-1}\right)^2} \quad =\frac{-2}{\frac{4}{(x-1)^2}} \quad =\frac{(x-1)^2}{-2} \quad =\frac{1}{\frac{dy}{dx}}$

39	$f(x)=\frac{\ln x}{x^2}$ $f'(x)=\frac{x^2\left(\frac{1}{x}\right)-(\ln x)(2x)}{x^4}=\frac{1-2\ln x}{x^3}$ $f''(x)=\frac{x^3\left(-\frac{2}{x}\right)-(1-2\ln x)(3x^2)}{x^6}$ $=\frac{-5x^2+6x^2\ln x}{x^6}$ $=\frac{-5+6\ln x}{x^4}$
40	$x^4f''(x)+4x^3f'(x)+2x^2f(x)+1$ $=x^4\times\frac{-5+6\ln x}{x^4}+4x^3\times\frac{1-2\ln x}{x^3}+2x^2\times\frac{\ln x}{x^2}+1$ $=-5+6\ln x+4-8\ln x+2\ln x+1=0$